ON THE EFFLUX OF A VISCOUS GAS INTO A VACUUM

(OB ISTECHENII VIAZKOGO GAZA V PUSTOTU)

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With the aid of the Navier-Stokes equations, the viscous flow from plane and spherical sources is studied under the assumption that the coefficient of viscosity depends on the temperature according to a power law, the Prandtl number being constant. The asymptotic solution is sought corresponding to efflux of gas into a vacuum when the pressure at infinity tends to zero.

In [1] were formulated conditions under which the viscosity and heat conduction had no effect on the asymptotic behaviour of the solution for a nonviscous supersonic source. In the present paper the case is investigated when these conditions are not fulfilled, which is generally true in practice.

It is shown that for a plane source the velocity at infinity tends to a value somewhat less than the corresponding maximum velocity for a nonviscous stream. For a spherical source an unexpected result is obtained: the velocity of the gas at infinity tends to zero.

Estimates derived in the paper show that in the region where the forces of viscosity in the momentum equations are comparable with the forces of inertia, the Navier-Stokes equations, generally speaking, lose their validity (just as in consideration of the structure of a shock wave). Nevertheless it can be expected that these equations, as in the case of a shock wave, give in a certain sense a correct qualitative description of the behavior of the flow.

1. Let us consider hypersonic flow of an ideal gas in a nozzle, the initial portion of which determines the flow in a certain region D extending to infinity [1,2]. Suppose that immediately behind the initial portion there is a sudden expansion of the stream, so that there takes place a free efflux of the gas into a vacuum (in practice - into a region with a pressure much lower than the pressure at the end of the initial portion of the nozzle). With such an efflux into a vacuum the influence of the dissipative processes, unaccompanied by losses of momentum and heat to the surroundings, becomes apparent, not as in a boundary layer, but rather as in the front of a shock wave [1].

Let us assume a power law dependence of the coefficient of viscosity μ on the temperature T ($\mu \sim T^n$). On the assumption that in the region D there is flow out of a source (in the general case the intensity of the source varies as we pass from one streamline to another [1,2]), the viscosity and the heat conduction do not change the asymptotic behavior of the flow at infinity when $n > n^\circ$ [1]. The quantity n° for plane and axisymmetric flow is, respectively

$$n^\circ = 1, \qquad n^\circ = 1 + \frac{1}{2(\varkappa - 1)}$$

where κ is the adiabatic exponent.

When $n < n^\circ$, at a certain distance r° from the centre of the source viscosity begins to make its presence felt, and the asymptotic expansion [1] found from Euler's equations becomes invalid. We shall find a solution for plane and spherical sources when $n < n^\circ$ in the region where the forces of viscosity are important.

2. Viscous flow from a plane source has been considered earlier in [3,4]. In [3] the problem was solved under the assumption that the coefficient of viscosity μ and heat conduction k are constant, whilst the Prandtl number σ has a certain fixed numerical value. In [4] the solution is constructed for two cases, when one of the coefficients, μ or k, is equal to zero. The results of [3,4] (when $k = 0$) are very close. The problem reduces to the study of an ordinary differential equation of the first order with the coefficient of the derivative equal to $1/R$, where R is the Reynolds number determined by the mass discharge Q of the source ($R \sim Q/\mu$). For a sufficiently large value of R the solution corresponds to the supersonic branch of nonviscous source flow. At a certain distance r from the center of the source a transition is possible in a narrow region of thickness R^{-1} to the subsonic branch, generally speaking, of another nonviscous source flow. Here the pressure tends at infinity to a certain constant value p_∞, different from zero. The position of this transition region, which is an ordinary shock wave formed by the influence of viscosity, is determined by the value of p_∞, just as the position of the density jump in a supersonic nozzle is determined by the given pressure at the outlet.

Moreover, there exists an integral curve, corresponding to the efflux of gas into a vacuum, the pressure along which tends to zero. In the particular case μ = const, $k = 0$ this solution is indicated in [4]. This solution will be worked out below for the case where μ and k depend on T according to a power law (n is the exponent), and the value of Prandtl's number σ is constant.

The equations of plane viscous radial flow, after eliminating the pressure and density by means of the equations of continuity and state, can be reduced to the following dimensionless form, as shown in [3]

$$(\varkappa w^2-\theta)\,w'+w\theta'+w\theta=-\frac{\varkappa w^2\theta^n}{R_*}(w''-w)-\frac{\varkappa n w^2\theta^{n-1}\theta'}{R_*}\left(w'+\frac{w}{2}\right) \quad (2.1)$$

$$\theta+\frac{\varkappa-1}{2}\left(1+\frac{\theta^n}{R_*}\right)w^2+\frac{\theta^n}{R_*}\left(\frac{3\theta'}{4\sigma}+\frac{\varkappa-1}{2}\frac{dw^2}{d\xi}\right)=\frac{\varkappa+1}{2} \quad (2.2)$$

$$\theta=\frac{T}{T_*},\quad T_*=\frac{2}{\varkappa+1}T_0,\quad w=\frac{V}{V_*},\quad V_*=\sqrt{\varkappa C T_*}$$
$$\xi=\ln\frac{r_*}{r},\quad R_*=\frac{4Q}{4\mu} \quad (2.3)$$

Here (2.1) is the equation of momentum, (2.2) is the energy integral, which exists in the case of radial flow. We have introduced the following notation: V is the gas velocity; w and θ are the dimensionless velocity and temperature, respectively; T_0 is the stagnation temperature determined at the point where the velocity and also the gradients of velocity and temperature are zero. T_* and V_* are the critical temperature and velocity, respectively, expressed in terms of T_0 by the formulas for the nonviscous flow; C is the gas constant; the coefficient of viscosity is given in the form $\mu = \mu_* (T/T_*)^n$; r is the distance from the center of the source; r_* is a characteristic length, which in what follows can be taken as the radius of the equivalent nonviscous source with the same value of heat content; R_* is the Reynolds number corresponding to the critical temperature; a dash denotes differentiation with respect to the independent variable ξ; the remaining notation has been introduced above.

Suppose that w and θ tend at infinity ($\xi \to -\infty$) to certain limiting values w_∞ and θ_∞, respectively. From the requirement that the quantities w_∞ and θ_∞ be not infinite, we have

$$\lim w'=0,\qquad \lim\theta'=0 \text{ when } \xi\to-\infty \quad (2.4)$$

We can show that when $n < n^0$ the quantity θ_∞ is not equal to zero. Assuming for the sake of argument that $\theta_\infty = 0$ and setting $w = w_\infty(1 + \Delta)$, where $|\Delta| \ll 1$, we find from Equation (2.2), taking into account (2.4), that the velocity at infinity tends to the maximum velocity V_m for the inviscid flow, i.e. $w_\infty^2 = (\kappa + 1)/(\kappa - 1)$. From Equation (2.3) we have, moreover, $|\Delta| \sim \theta^n$ when $n < n^\circ = 1$ and $|\Delta| \sim \theta$ when $n > 1$. Making use of the expression for Δ and ignoring small quantities we reduce Equation (2.1) to a differential equation of the first order. It turns out that when $n < 1$ a solution for which θ tends to zero at infinity does not exist. When $n > 1$ the solution obtained for inviscid flow is valid.

We are left to conclude that $\theta_\infty \neq 0$ when $n < n^\circ$.

To determine the quantities w_∞ and θ_∞, taking account of (2.4), we have from Equations (2.1) and (2.2)

$$w_\infty\theta_\infty = \frac{\varkappa w_\infty{}^3\theta_\infty{}^n}{R_*}, \qquad \theta_\infty + \frac{\varkappa-1}{2}\left(1+\frac{\theta_\infty{}^n}{R_*}\right)w_\infty{}^2 = \frac{\varkappa+1}{2} \tag{2.5}$$

In obtaining the first Equation (2.5) we have made use of the condition $w'' \to 0$ when $\xi \to -\infty$, following from the fact that w' tends to zero (2.4). From Equations (2.5) we can obtain a transcendental algebraic equation for determining θ_∞, having solved which, we can find w_∞ as well:

$$\frac{3\varkappa-1}{\varkappa(\varkappa+1)}\theta_\infty + \frac{\varkappa-1}{\varkappa(\varkappa+1)}R_*\theta_\infty{}^{1-n} = 1, \qquad w_\infty{}^2 = \frac{R_*}{\varkappa}\theta_\infty{}^{1-n} \tag{2.6}$$

Let us assume that the Reynolds number R_* is sufficiently large. Then, solving the first Equation (2.6) by the method of successive approximations and limiting ourselves to two approximations, we have

$$\theta_\infty = \left(\frac{\alpha}{R_*}\right)^{\frac{1}{1-n}}\left[1-\frac{\beta}{1-n}\left(\frac{\alpha}{R_*}\right)^{\frac{1}{1-n}}\right], \qquad \alpha = \frac{\varkappa(\varkappa+1)}{\varkappa-1}, \qquad \beta = \frac{3\varkappa-1}{\varkappa(\varkappa+1)}$$

$$w_\infty = \sqrt{\frac{\varkappa+1}{\varkappa-1}}\left[1-\frac{\beta}{2}\left(\frac{\alpha}{R_*}\right)^{\frac{1}{1-n}}\right] \tag{2.7}$$

For the limiting Mach number M_∞ we obtain, accepting the first approximation

$$M_\infty = \frac{w_\infty}{\sqrt{\theta_\infty}} = \sqrt{\frac{\varkappa+1}{\varkappa-1}}\left(\frac{R_*}{\alpha}\right)^{\frac{1}{2(1-n)}} \tag{2.8}$$

Accordingly, the Mach number in plane flow efflux into a vacuum tends to a finite limit as a result of the influence of the dissipative processes, whilst the velocity of the gas tends to a value somewhat less than V_m.

As has already been remarked, the Expressions (2.7) can be used when $n < 1$. When $n \geqslant 1$ we have, according to [1], $w_\infty{}^2 = (\kappa + 1)/(\kappa - 1)$, $\theta_\infty = 0$.

To determine the asymptotic character of the behavior of the solution, let us write the required functions in the form $w = w_\infty(1 + \Delta)$, $\theta = \theta_\infty(1 + \eta)$, where Δ and η are small quantities, in Equations (2.1) and (2.2). Let us find the solution in stages, first of all to an accuracy of the first order in Δ and η, then to an accuracy of the second order, and so on. As a result we obtain

$$w = w_\infty(1 + a_1 e^{k\xi} + a_2 e^{2k\xi} + \ldots) = w_\infty\left[1 + a_1\left(\frac{r_*}{r}\right)^k + a_2\left(\frac{r_*}{r}\right)^{2k} + \ldots\right]$$

$$\theta = \theta_\infty(1 + b_1 e^{k\xi} + b_2 e^{2k\xi} + \ldots) = \theta_\infty\left[1 + b_1\left(\frac{r_*}{r}\right)^k + b_2\left(\frac{r_*}{r}\right)^{2k} + \ldots\right] \quad (2.9)$$

The coefficients of these series depend on one parameter, which we can take to be, for example, the quantity $b_1 r_*^{\,k}$. It is convenient, however, to choose r_* equal to the critical radius of the inviscid source, flow from which approximates to the flow from the viscous source under consideration in the region where the viscous terms are small in comparison with the convective terms. Then for the arbitrary parameter we take the quantity b_1. We have the following equations:

$$a_n = b_1^{\,n} f_n\,(k,\ \theta_\infty,\ w_\infty), \qquad b_n = b_1^{\,n}\varphi_n\,(k,\ \theta_\infty,\ w_\infty)$$

where f_n and φ_n are known functions. For example, f_1 is expressed by

$$f_1 = \theta_\infty \frac{n(1 - 0.5k) - k - 1}{\varkappa k w_\infty^{\,2} + \theta_\infty(k^2 - k - 2)} \quad (2.10)$$

The parameter k in (2.9) and (2.10) is determined from the equation

$$\frac{3\lambda^2}{4\varkappa\sigma}k^3 + \lambda\left[\frac{1}{\varkappa} + \frac{3}{4\sigma}\left(1 - \frac{\lambda}{\varkappa}\right)\right]k^2 + \left[1 + \left(\frac{2}{\varkappa} - 3\right)\lambda - \frac{6\lambda^2}{4\varkappa\sigma}\right]k +$$
$$+\left[\left(-3 + \frac{1}{\varkappa}\right)\lambda + (\varkappa - 1)(n - 1)\right] = 0 \qquad \left(\lambda = \frac{\theta_\infty}{w_\infty^{\,2}}\right) \quad (2.11)$$

It is necessary to choose the root of this equation which remains finite as R_* tends to infinity, i.e. as $\lambda \to 0$. For this root we obtain an approximate expression from (2.11):

$$k = k_0 + \lambda\left[3 - \frac{1}{\varkappa} - \frac{2 - 3\varkappa}{\varkappa}k_0 - \frac{k_0^{\,2}}{\varkappa}\right] + O(\lambda^2), \qquad k_0 = (\varkappa - 1)(1 - n) \quad (2.12)$$

Using the equations of continuity and state, it is easy to perceive that for the solution under consideration the pressure tends to zero.

3. Let us consider viscous flow from a spherical source. In [5] this investigation was carried out under the assumption that r does not exceed a certain fixed value r_1, and for constant values of the coefficients of viscosity and heat conduction. Under these conditions results were obtained similar to the case of a plane source. The flow up to a certain value $r = r_0$ corresponds to the supersonic branch of an inviscid source flow, and then passes through a shock wave to the subsonic branch of, generally speaking, another source flow. The pressure at infinity for the subsonic source, when not equal to zero, determines the position of the shock wave.

Let us study the asymptotic behavior of the solution for which the pressure at infinity tends to zero. As in the first section, we shall assume a power law for the nature of the dependence of the coefficients of viscosity and heat conduction upon temperature. The equations of motion can be reduced, after eliminating pressure and density, to the form

$$(\varkappa w^2 - \theta)\, w' + w\theta' + \frac{2w\theta}{y} = -\frac{\varkappa w^2\theta^n}{R_*}\left(w'' - \frac{2w}{y^2}\right) - \frac{n\varkappa w^2\theta^{n-1}\theta'}{R_*}\left(w' + \frac{2w}{y}\right) \qquad (3.1)$$

$$\theta + \frac{\varkappa - 1}{2}\left(1 + \frac{2\theta^n}{R_* y}\right) w^2 + \frac{\theta^n}{R_*}\left(\frac{3\theta'}{4\sigma} + \frac{\varkappa - 1}{2}\frac{dw^2}{dy}\right) = \frac{\varkappa + 1}{2} \qquad (3.2)$$

$$(y = r_* / r, \quad R_* = 3Q / 4\mu r_*)$$

Here the rest of the notation is the same as in Section 1.

It turns out that, in contrast to the case of the plane source with $n < n^{\circ} = 1 + 0.5(\kappa - 1)^{-1}$, the velocity w obtained from the solution of Equations (3.1) and (3.2) tends to zero at infinity. We shall present a proof for $n = 0$. Let us suppose on the contrary, that w tends to $w_\infty > 0$ at infinity. Let us integrate Equation (3.2) with respect to y, denoting by the index 1 the values of quantities when $y = y_1$. As a result we obtain

$$\int_{y_1}^{y} \theta\, dy + \frac{\varkappa - 1}{R_*}\int_{y_1}^{y} w^2 \frac{dy}{y} + \frac{3(\theta - \theta_1)}{4\sigma R_*} =$$

$$= -\frac{\varkappa - 1}{2}\int_{y_1}^{y} w^2\, dy - \frac{\varkappa - 1}{2}(w^2 - w_1^2) + \frac{\varkappa + 1}{2}(y - y_1) \qquad (3.3)$$

The right-hand side of Equation (3.3) remains finite as y tends to zero (which corresponds to $r \to \infty$). Hence it follows that the left-hand side of the equation must also remain finite, i.e. θ must satisfy the asymptotic expression

$$\theta \approx -\frac{4\sigma(\varkappa - 1)}{3} w_\infty^2 \ln y \quad \text{when } y \to 0 \qquad (3.4)$$

Equation (3.4) contradicts the physical sense of the problem, since it follows thence that the temperature of the gas increases without limit at infinity. We shall show that the assumption $w_\infty > 0$ contradicts also the first Equation (3.1). Let us rewrite this equation with $n = 0$ in the form

$$\frac{w''}{R_*} + w' - \frac{2w}{R_* y^2} + \frac{1}{\varkappa}\left[\frac{d}{dy}\left(\frac{\theta}{w}\right) + \frac{2\theta}{wy}\right] = 0 \tag{3.5}$$

Let us integrate Equation (3.5) twice in the interval from a certain fixed value $y = y_1$ up to y. Eventually we obtain

$$\frac{w}{R_*} = \frac{w_1}{R_*} + \frac{w'_1(y-y_1)}{R_*} - \int_{y_1}^{y} w\,dy + (y-y_1)\,w_1 + \frac{1}{\varkappa}\int_{y_1}^{y}\frac{\theta}{w}\,dy +$$
$$+ \frac{(y-y_1)\,\theta_1}{\varkappa w_1} - \frac{2y}{\varkappa}\int_{y_1}^{y}\frac{\theta}{w}\,\frac{dy}{y} + \frac{2y}{R_*}\int_{y_1}^{y}\frac{w}{y^2}\,dy - \frac{2}{R_*}\int_{y_1}^{y}\frac{w}{y}\,dy \tag{3.6}$$

Now let y tend to zero. Taking account of (3.4) and assuming that w is finite, we find that all the terms on the right-hand side of (3.6) remain finite except the last, which is expressed asymptotically in the form $(2w_\infty/R_*)\ \ln\,(1/y)$. At the same time the left-hand side of the equation remains finite. We have therefore arrived at a contradiction of our initial assumption, according to which $w_\infty > 0$. We are left to conclude that the velocity of the gas at infinity tends to zero.

A similar but more complicated proof can be presented also for $0 < n < n^\circ$. Let us trace the course of the proof. Assuming that a solution of the formulated problem exists, i.e. the quantities w and θ do not tend anywhere to infinity, let us integrate Equation (3.2) with respect to y when $n \neq 0$. As a result we find that the integral

$$J = \int_{y_1}^{y}\frac{\theta^n w^2}{y}\,dy \tag{3.7}$$

must converge with y tending to zero. Hence it follows that either the temperature or the velocity for the spherical source must vanish at infinity. When $n > n^\circ$ the temperature tends to zero, whilst the velocity tends to the maximum velocity [1] for the nonviscous source V_*. When $n < n^\circ$ the temperature θ_∞ is different from zero, whilst the velocity $w_\infty = 0$. Assuming the opposite, that $w = w_\infty(1 + \Delta)$, where $|\Delta| \ll 1$ and $w_\infty \neq 0$, let us substitute the expression for w in Equations (3.1) and (3.2). Investigating the system so obtained, we are forced to accept the truth of the stated theorem.

The solution in the neighborhood of a point at infinity, for which the pressure tends to zero, has the following form when $n < n^{\circ}$:

$$w = \sqrt{y}\,(a_0 + a_1 y + a_2 y^2 + \ldots), \qquad \theta = b_0 + b_1 y + b_2 y^2 + \ldots \tag{3.8}$$

The coefficients of these series depend on a parameter, which we can take to be, for example, the quantity b_0

$$a_0^2 = \frac{2R_*}{3\varkappa} b_0^{1-n}, \qquad b_1 = \frac{2\sigma(\varkappa+1)R_*}{3b_0^{\,n}}\left[1 - \frac{2(2\varkappa-1)}{\varkappa(\varkappa+1)} b_0\right] \tag{3.9}$$

The quantity r_*, just as in Section 1, is chosen equal to the critical radius of the nonviscous source, flow from which approximates to the flow from the viscous source under consideration in the region where the forces of viscosity are unimportant.

4. The results obtained indicate the fact that the dissipative processes for plane and spherical sources when $n < n^{\circ}$ (and consequently, also for the plane and axisymmetric cases of efflux into a vacuum) prove to be essentially different. In the first case the velocity at infinity differs only slightly from the maximum velocity V_m for a nonviscous flow, whereas in the second case the velocity of the gas tends to zero. These results are obtained under the assumption that the Navier-Stokes equations are valid. However, starting at a certain distance, these equations lose their validity. In order to determine the boundary of the applicability of the Navier-Stokes equations, let us find that region in which the ratio of the supplementary terms in Burnett's equations to the terms in the Navier-Stokes equations representing the effect of viscosity, becomes a quantity of order unity. This is in fact the criterion that the Navier-Stokes equations lose their validity. The indicated ratio ψ is (see, for example, [6])

$$\psi = \frac{\mu}{p}\,\mathrm{div}\,V \sim \frac{\mu V}{pr} \tag{4.1}$$

where p is the pressure, and the rest of the notation is the same as in Section 1. Making use of the asymptotic representation of the solution for a nonviscous source [1], we find from Equation (4.1) that

$$\psi = \frac{3\varkappa}{4R_*}\left(\frac{\varkappa-1}{\varkappa+1}\right)^h \left(\frac{r}{r_*}\right)^s, \qquad h = \frac{(\varkappa-1)(n-1)}{2}$$

$$s = (\varkappa-1)(1+\nu)(1-n) + \nu \tag{4.2}$$

where $\nu = 0$ and 1, respectively, for plane and spherical sources. Let us now write down, however, the ratio φ of the viscous terms in the Navier-Stokes equations to the convective terms. This ratio, according to [1], is

$$\varphi = \frac{3}{4(\nu+1)R_*}\left(\frac{\varkappa-1}{\varkappa+1}\right)^q\left(\frac{r}{r_*}\right)^s, \qquad q = \frac{(\varkappa-1)n-(\varkappa+1)}{2} \tag{4.3}$$

where s has the same value as in Equation (4.2). Comparing (4.2) and (4.3), we find that

$$\frac{\psi}{\varphi} = \frac{\varkappa(\varkappa-1)(1+\nu)}{\varkappa+1} \tag{4.4}$$

From (4.4) it follows that in a region where the viscous terms of the Navier-Stokes equations are comparable in order of magnitude with the nonviscous terms ($\varphi \sim 1$), the Burnett terms are of the same order as the viscous terms, since $\psi \sim \varphi$. In other words, in a region where dissipation is important, the Navier-Stokes equations are, generally speaking, inapplicable. The question arises as to how much the results obtained in Sections 2 and 3 of this paper on the basis of an analysis of the Navier-Stokes equations, agree with the truth. This question remains as yet unanswered. We can, however, think that, as in the study of the structure of a shock wave, the Navier-Stokes equations give a correct qualitative description of the flow pattern up to such times as the flow begins to approximate to free molecule flow.

The results obtained can be considered as an indication of the circumstance that the velocity of free-molecular flow for plane efflux is close to V_m, whereas for three-dimensional flow this velocity can be significantly less than V_m.

On the other hand, it follows from the law of conservation of energy that the flow of total energy from the source in unit time, equal to $QV_m^2/2$, must be equal to the flow of energy of free-molecular flow, which consists of the energy of its orderly radial motion and the energy of the random motion, corresponding to the external (progressive motion) and the internal (rotational, vibrational motion and so on) degrees of freedom. Since the mass discharge for free-molecular flow through a closed surface enclosing the source is equal to the intensity of the source Q, then we can write down the equation

$$V_m^2 = V_\infty^2 + c^2 \tag{4.5}$$

where V_∞ is the radial velocity of the macroscopic motion, whilst c is a quantity with the dimensions of velocity, the square of which is equal to twice the energy of the averaged random motion of the molecules in unit mass. The quantity c evidently characterizes the "thermal scattering" of free-molecular motion.

From (4.5) and the foregoing considerations it follows that in plane flow the thermal scattering is not large, i.e. nearly all the energy of the source passes into energy of orderly motion. A different pattern emerges in three-dimensional flow: as a result of the strong influence of the dissipative processes a significant part of the energy of the stream can be transformed into energy of random thermal motion of the molecules.

We note that a similar phenomenon can occur in problems of unsteady motion of a gas dispersing into a vacuum. Here also, it appears, a very important part is played by the dissipative processes.

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